Inflationary models and the recent observations

Main reference: Phys. Rev. D 90, 023525 (2014) [arXiv:1404.4311 [gr-qc]]

2nd International Workshop on Particle Physics and Cosmology after Higgs and Planck

NCTS, Hsinchu and Fo-Guang-Shan, Kaohsiung, Taiwan

11th October, 2014

Presenter: **Kazuharu Bamba** (*LGSPC, Ochanomizu University*)

Collaborators: **Guido Cognola** (*Dep. of Phys., Trento University*)

Sergei D. Odintsov (*ICE/CSIC-IEEC and ICREA*)

Sergio Zerbini (*Dep. of Phys., Trento University*)

Contents

I. Introduction

Planck satellite

- **Spectral index of power spectrum of the curvature perturbations**

$$n_{\mathrm{s}} = 0.9603 \pm 0.0073 \, (95\% \, \mathrm{CL})$$

- **Running of the spectral index**

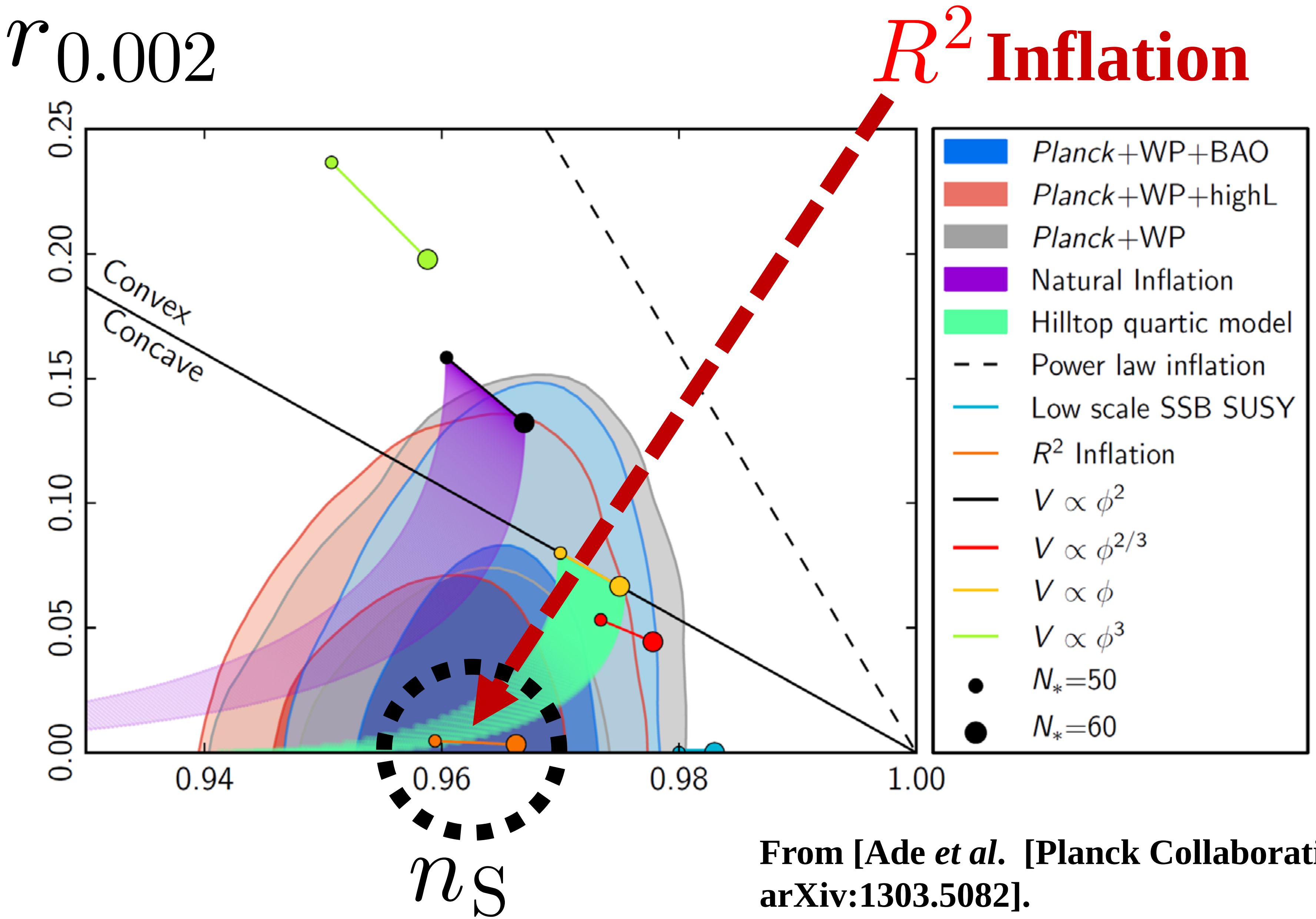
$$\alpha_{\mathrm{s}} = -0.0134 \pm 0.0090 \, (68\% \, \mathrm{CL})$$

- **Tensor-to-scalar ratio**

$$r < 0.11 \, (95\% \, \mathrm{CL})$$

[Ade *et al*. [Planck Collaboration], arXiv:1303.5076; arXiv:1303.5082]

Planck results



From [Ade *et al*. [Planck Collaboration], arXiv:1303.5082].

BICEP2 experiment

$$r = 0.20^{+0.07}_{-0.05} \, (68\% \, \mathrm{CL})$$

[Ade *et al*. [BICEP2 Collaboration], Phys. Rev. Lett. 112, 241101 (2014)]

Cf. [Ade *et al*. [Planck Collaboration], arXiv:1405.0871 [astro-ph.GA]]

[Ade *et al*. [Planck Collaboration], arXiv:1405.0874 [astro-ph.GA]]

[Adam *et al*. [Planck Collaboration], arXiv:1409.5738 [astro-ph.CO]]

Motivation

- **Various modified gravity theories have recently been proposed to explain cosmic acceleration.**

Inflation ⟵ R^2 **gravity**

[Starobinsky, Phys. Lett. B 91, 99 (1980)]

Dark Energy problem ⟵ $f(R)$ **gravity**

[Capozziello, Carloni and Troisi, Recent Res. Dev. Astron. Astrophys. 1, 625 (2003)]

[Nojiri and Odintsov, Phys. Rev. D 68, 123512 (2003)]

[Carroll, Duvvuri, Trodden and Turner, Phys. Rev. D 70, 043528 (2004)]

Purpose

- **We compare the nature of classical expressions of modified gravity with that with quantum corrections.**

⇩

We investigate a generalized model whose Lagrangian is described by a function of $f(R, K, \phi)$.

R : Scalar curvature

ϕ : Scalar field

K : Kinetic term of ϕ

Purpose (2)

- We show that in the Jordan and Einstein frames, $f(R)$ gravity is equivalent in the quatum level.

- We discuss the stability of the de Sitter solutions and explore the influence of the one-loop quantum correction on inflation in R^2 gravity with the quantum correction.

Cf. [Cognola, Elizalde, Nojiri, Odintsov and Zerbini, JCAP 0502, 010 (2005)]

II. Model

Action $I = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} f(\tilde{R}, \tilde{K}, \tilde{\phi})$ $\kappa^2 = 8\pi G$

$$\tilde{K} = (1/2)\, \tilde{g}^{ij} \tilde{\nabla}_i \tilde{\phi} \tilde{\nabla}_j \tilde{\phi}$$

G : Gravitatiotanl constant

$\tilde{\nabla}_i$: Covariant derivative

$\tilde{\Delta}$: Laplacian

* The tilde denotes the quantities in the Einstein frame.

Gravitational field equation

$$f_{\tilde{R}} \tilde{R}_{ij} - \frac{1}{2} f \,\tilde{g}_{ij} + \left(\tilde{g}_{ij} \tilde{\Delta} - \tilde{\nabla}_i \tilde{\nabla}_j\right) f_{\tilde{R}} + \frac{1}{2} f_{\tilde{K}} \tilde{\nabla}_i \tilde{\phi} \tilde{\nabla}_j \tilde{\phi} = 0$$

Equation of motion for $\tilde{\phi}$

$$\tilde{g}^{ij} \tilde{\nabla}_i \left(f_{\tilde{K}} \tilde{\phi} \tilde{\nabla}_j \phi\right) = f_{\tilde{\phi}}$$

$$f_{\tilde{R}} \equiv \frac{\partial f}{\partial \tilde{R}}$$

Solutions for the equations of motion

- **There is a constant curvature solution:**

$$\tilde{R} = R$$

- **For** $\tilde{\phi} = \phi = \text{constant}$**,**

$$\longrightarrow \quad f_R\, R_{ij} - \frac{1}{2} f_0\, g_{ij} = 0\,, \qquad f_\phi = 0$$

$$f_0 = f(R, K, \phi)$$

⇨ **The set of background fields (constant curvature, constant scalar field) is a solution of the following equations:**

$$R\, f_R - 2 f_0 = 0\,, \qquad f_\phi = 0$$

III. Quantum equivalence

- **Modified gravity: Described in the Jordan frame**

$\rightarrow$ $f(R)$ **gravity: Can also be described in the Einstein frame**

These are equivalent in the classical level.

[Maeda, Phys. Rev. D 39, 3159 (1989)]

[Fujii and Maeda, *The Scalar-Tensor Theory of Gravitation* (2003)]

⇨ **We show the on-shell quantum equivalence of $f(R)$ gravity.**

[Buchbinder, Odintsov and Shapiro, *Effective action in quantum gravity* (1992)]

[Fradkin and Tseytlin, Nucl. Phys. B234, 472 (1984)]

Quantum equivalence (2)

$$I_{\rm Jord} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, f(R)$$: **Jordan frame**

$$\tilde{g}_{ij} = e^{\sigma} g_{ij}$$: **Conformal transformation**

$$I_{\rm Eins} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}}\, \tilde{f}(\tilde{R}, K, \sigma)$$: **Einstein frame**

$$\tilde{f}(\tilde{R}, \tilde{K}, \sigma) = \tilde{R} - \frac{3}{2} \tilde{g}^{ij}\, \partial_i \sigma \partial_j \sigma - V(\sigma)$$

$$e^{\sigma} = f'(R), \qquad R = \Phi(e^{\sigma}), \qquad \Phi \circ f' = 1$$

$$V(\sigma) \equiv e^{-\sigma} \Phi(e^{\sigma}) - e^{-2\sigma} f(\Phi(e^{\sigma})) \qquad f' \equiv \frac{\partial f}{\partial R}$$

Quantum equivalence (3)

Jordan frame

Contribution of scalars to the effective action

$$\Gamma^{\mathrm{Jord}}_{\mathrm{on-shell}} = \frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-f_{RR}\left(\Delta_0 + \frac{R}{3}\right) + \frac{f_R}{3}\right)\right]$$

$$+\text{classical and higher spin contributions}$$

Δ_0 : Laplacian acting on scalars

μ^2 : Renormalization parameter

Quantum equivalence (4)

Einstein frame

$$V''(\sigma) \equiv \frac{\partial^2 V(\sigma)}{\partial \sigma^2}$$

$$\Gamma^{\rm Eins}_{\rm on-shell} = \frac{1}{2}\ln\det\left[\frac{1}{\tilde{\mu}^2}\left(3\tilde{\Delta}_0 - V''(\sigma)\right)\right]$$

$$+\text{classical and higher spin contributions}$$

$$= \frac{1}{2}\ln\det\left[\frac{1}{\tilde{\mu}^2}\left(\frac{3\Delta_0}{f_R} + \frac{R}{f_R} - \frac{1}{f_{RR}}\right)\right]$$

$$+\text{classical and higher spin contributions}$$

By the redefinition of $\tilde{\mu}$, this can become equivalent to $\Gamma^{\rm Jord}_{\rm on-shell}$.

R^2 gravity

Jordan frame

M^2 : Mass parameter

$$f(R) = R + \frac{R^2}{6M^2}$$

[Starobinsky, Phys. Lett. B 91, 99 (1980)]

$$\Gamma^{\text{Jord}}_{\text{on-shell}} = \frac{1}{2} \ln \det \left[\frac{1}{\mu^2} \left(-\Delta_0 + M^2 \right) \right]$$

$+$classical and higher spin contributions

R^2 gravity (2)

Einstein frame

$$\tilde{f}(\tilde{R}, \tilde{K}, \sigma) = \tilde{R} - \frac{3}{2}\tilde{g}^{ij}\partial_i\sigma\partial_j\sigma - \frac{3}{2}M^2\left(1 - e^{-\sigma}\right)^2$$
$$= \tilde{R} - 3\tilde{K} - \frac{3}{2}M^2\left(1 - e^{-\sigma}\right)^2$$

(Cf. $\sigma \to \sqrt{2/3}\phi/M_{\rm P}$)

$$\Gamma^{(1)}_{\rm on-shell} = \frac{\mathcal{V}}{2}M^4\left(\ln\frac{M^2}{\mu^2} - \frac{3}{2}\right)$$

$\mathcal{V}$: Volume

R^2 gravity (3)

Jordan frame

$M^2/R \ll 1$ **: During inflation**

$$L(R) = \frac{1}{2} M_{\mathrm{P}}^2 \left[R + \frac{R^2}{6M^2} + \frac{R^2}{384\pi^2 M_{\mathrm{P}}^2} \left(C_1 \ln \frac{R}{\mu^2} + C_2 \right) \right] + O\left(\frac{M^2}{R} \right)$$

$$C_1 = O(1), \qquad C_2 \sim 300$$

Conformal transformation

$$\tilde{g}_{ij} = e^{\sigma} g_{ij}$$

M_{P} : Reduced Planck mass

R^2 gravity (4)

Einstein frame

$$I_{\text{Eins}} = \frac{1}{\kappa^2} \int d^4x \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{3}{2} \tilde{g}^{ij} \partial_i \sigma \partial_j \sigma - V(\sigma) \right)$$

$$V(\sigma) = \left(1 - e^{-\sigma}\right)^2 \frac{a + 2b \left\{ 1 + \log |e^{\sigma} - 1| - \log \left[4|b\mu| W \left(\frac{|e^{\sigma}-1| e^{(a+b)/2b}}{4|b\mu|} \right) \right] \right\}}{\left[4bW \left(\frac{|e^{\sigma}-1|] e^{(1+b)/2b}}{4|b\mu|} \right) \right]^2}$$

W : Lambert function

$$a = \frac{1}{6M^2} + \frac{C_2}{384\pi^2 M_P^2}, \qquad b = \frac{C_1}{384\pi^2 M_P^2}$$

R^2 gravity (5)

Jordan frame

$M^2/R \gg 1$ **: At the end of inflation**

$$L(R) = \frac{1}{2} M_P^2 \left[R + \frac{R^2}{6M^2} - \frac{M^4}{32\pi^2 M_{\rm P}^2} \left(\ln \frac{M^2}{\mu^2} - \frac{3}{2} \right) + O\left(R^2 \ln \frac{R}{M_{\rm P}^2 \mu^2} \right) \right]$$

$$\Lambda(\mu) = \frac{M^4}{16\pi^2 M_{\rm P}^2} \left(\ln \frac{M^2}{\mu^2} - \frac{3}{2} \right)$$

: Effective cosmological constant

R^2 gravity (6)

Einstein frame: R + Scalar field theory for ϕ

ϕ : Inflaton field

$$V(\phi) = \frac{1}{2} M_{\mathrm{P}}^2 \left[\frac{3M^2}{2} \left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\phi}{M_{\mathrm{P}}}} \right)^2 + 2\Lambda(\mu)\, e^{-2\sqrt{\frac{2}{3}} \frac{\phi}{M_{\mathrm{P}}}} \right]$$

$V(\phi)$ **becomes the minimum at** $\phi = \phi_*$.

$\phi / M_{\mathrm{P}} \gg 1$ **:**

R^2 **Inflation**

$$\Rightarrow \quad V(\phi_*) = \frac{3M_{\mathrm{P}}^2 M^2 \Lambda(\mu)}{3m^2 + 4\Lambda(\mu)}$$

Contribution of the quantum correction : $\mathcal{O}((M/M_{\mathrm{P}})^2)$

IV. Stability issue

The one-loop on-shell effective action

$$\Gamma^{(1)}_{\text{on-shell}} = \underline{\frac{1}{2}\ln\det\left[\frac{1}{\mu^4}\left(a_2\Delta_0^2 + a_1\Delta_0 + a_0\right)\right]}$$

$$-\frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_1 - \frac{R}{4}\right)\right] + \frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_2 + \frac{R}{6}\right)\right]$$

$$\begin{aligned} a_0 &= f_R\left[Rf_{R\phi}^2 + f_{\phi\phi}(f_R - Rf_{RR})\right] \\ a_1 &= f_R\left[f_K(Rf_{RR} - f_R) + f_{R\phi}^2 - 3f_{\phi\phi}f_{RR}\right] \\ a_2 &= 3f_K f_R f_{RR} \end{aligned}$$

Δ_1 : Laplacian acting on transverse-traceless vectors

Δ_2 : Laplacian acting on transverse-traceless tensors

Stability issue (2)

Stability condition:

All of the eigen values for the operators in $\Gamma^{(1)}_{\mathrm{on-shell}}$ are not negative.

Minimum eigen values of $\Delta_0,\ \Delta_1,\ \Delta_2$:

$\longrightarrow$ **The minimum eigen value of Δ_0 is the smallest one.**

⇨ **The first term of $\Gamma^{(1)}_{\mathrm{on-shell}}$ including Δ_0 is related to the stability.**

Stability issue (3)

$$f(R, K, \phi) = F(R) - \frac{1}{2} g^{ij}\, \partial_i \phi \partial_j \phi = F(R) - K$$

$$M_P^2/2 = 1$$

Stability condition

a_1/a_2 **:Non-negative value**

$$\Longrightarrow \quad \frac{F_R}{F_{RR}} - R \geq 0$$

Stability issue (4)

$$f(R,K,\phi) = R - \frac{1}{2} g^{ij}\, \partial_i\phi\partial_j\phi - \frac{1}{2} m^2\phi^2 + \xi R\phi^2$$

$m, \ \xi$: Constant

- **Background solution:** $R = \dfrac{m^2}{2\xi}\,, \quad \phi = \pm\dfrac{1}{\sqrt{\xi}}$

→ **For** $\xi \neq -1/6$ **,**

Stability condition

$$-\frac{a_0}{a_1} = -\frac{m^2}{1+6\xi} \geq 0 \quad \Longrightarrow \quad \xi < -\frac{1}{6}$$

V. Conclusions

- **We have studied a generalized model whose Lagrangian is described by a function of** $f(R, K, \phi)$ **.**

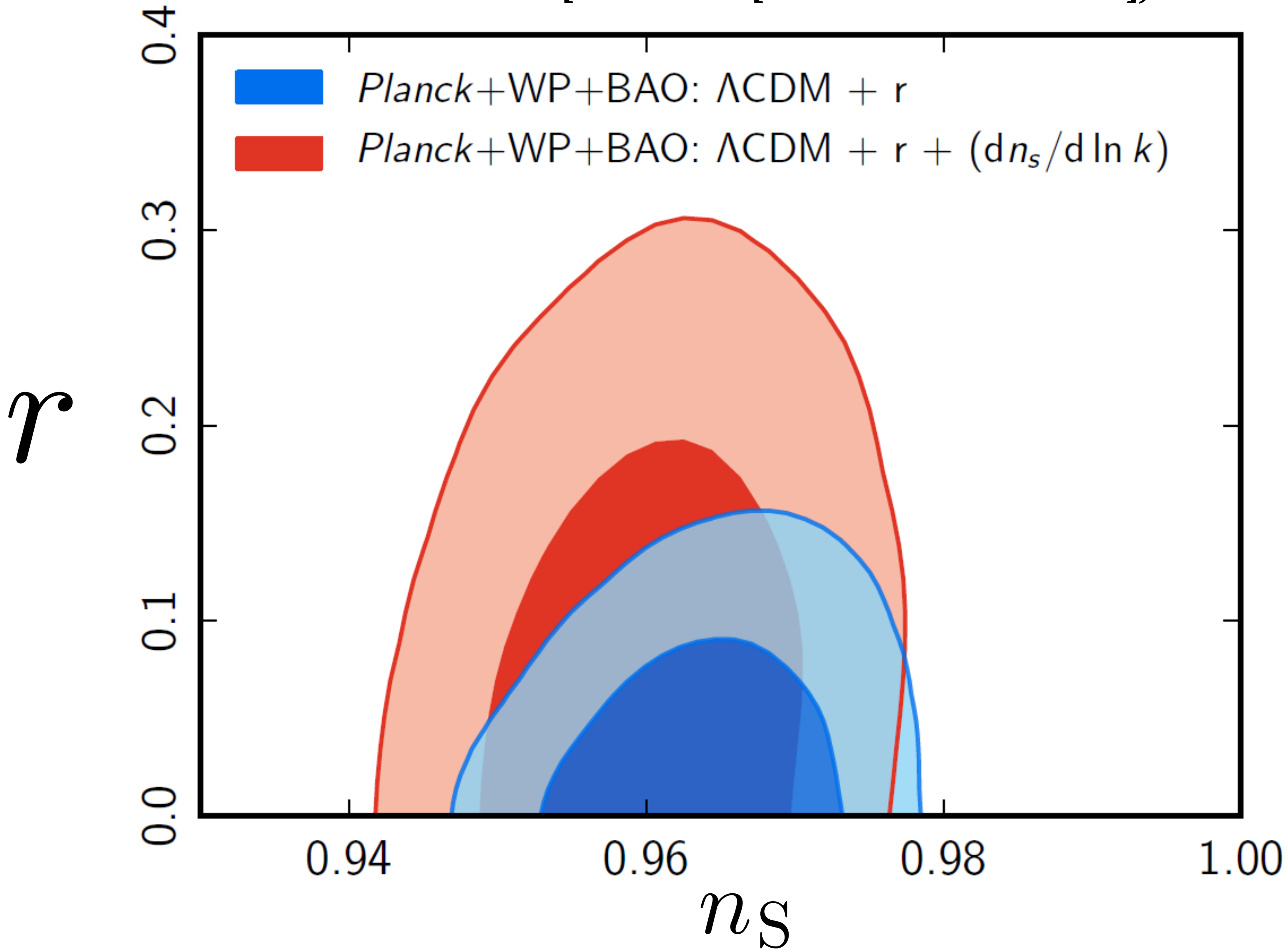
- **We have shown the on-shell quantum equivalence of** $f(R)$ **gravity in the Jordan and Einstein frames.**

- **We have examined the stability of the de Sitter solutions and the one-loop quantum correction to inflation in quantum-corrected** R^2 **gravity.**

Back up slides

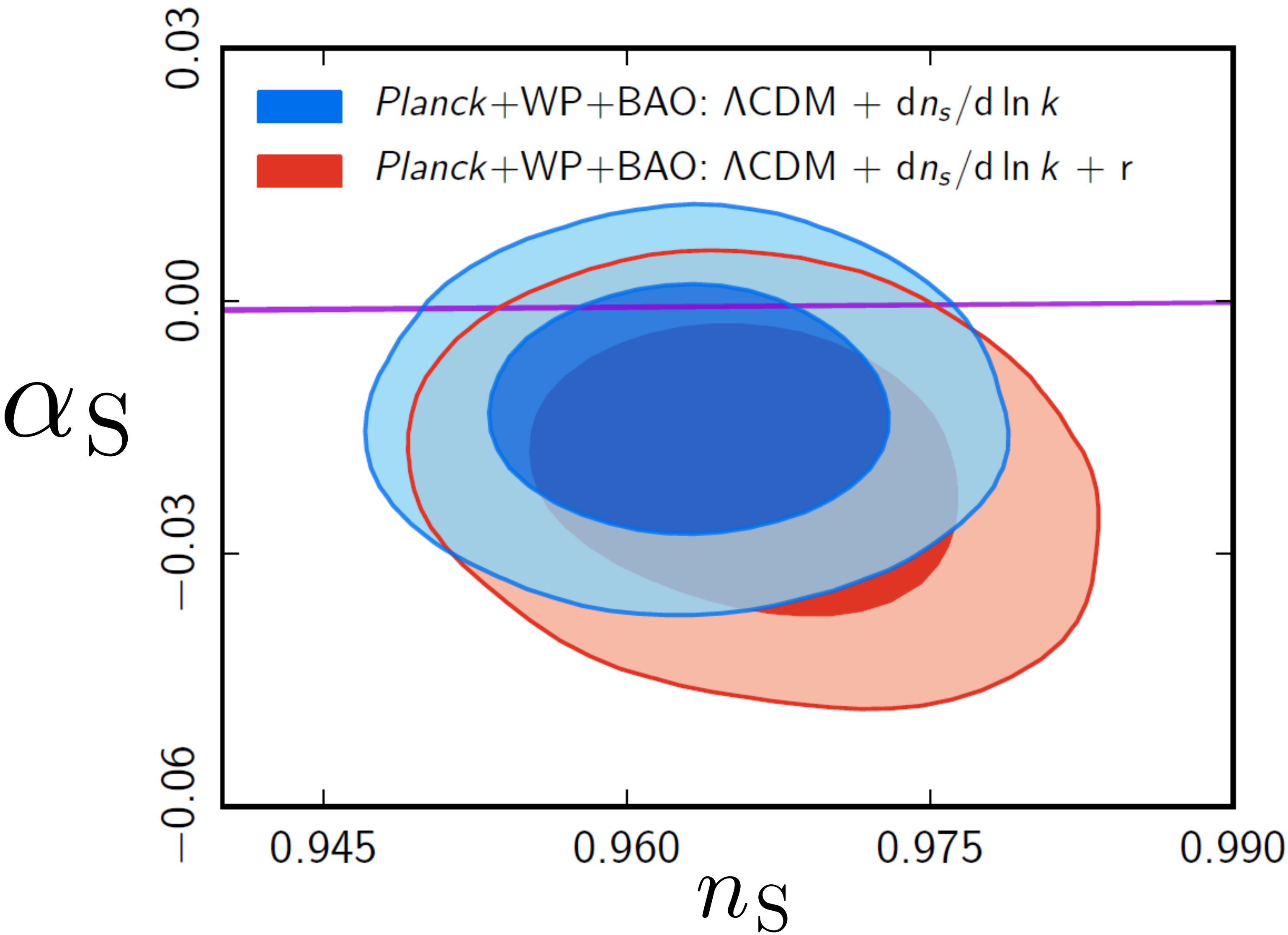
Planck satellite (2)

From [Ade *et al.* [Planck Collaboration], arXiv:1303.5082].



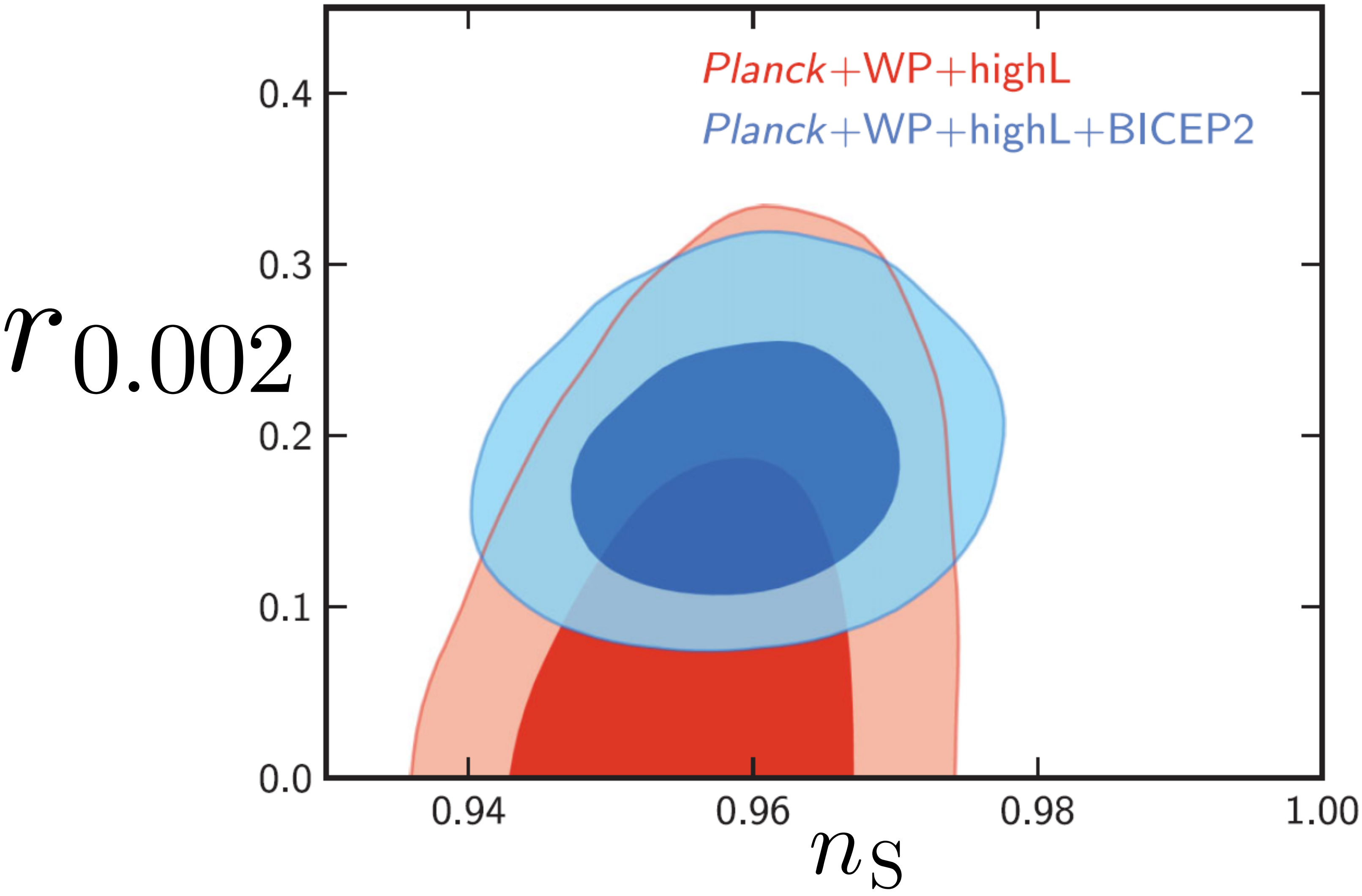
Planck satellite (3)

From [Ade *et al.* [Planck Collaboration], arXiv:1303.5082].



BICEP2 experiment (2)

[Ade *et al*. [BICEP2 Collaboration], Phys. Rev. Lett. 112, 241101 (2014)]



Inflation

$$\varepsilon = \left(\frac{1}{V}\frac{dV}{d\phi}\right)^2 = \frac{1}{3}\left(\frac{V'(\sigma)}{V(\sigma)}\right)^2, \qquad \eta = \frac{2}{V}\frac{d^2V}{d\phi^2} = \frac{2}{3}\frac{V''(\sigma)}{V(\sigma)}$$

$$n_{\mathrm{S}} = 1 - 6\epsilon + 2\eta, \quad r = 16\epsilon$$

$$M \sim 0.1\, M_P, \qquad \mu \sim M$$

$$\phi_k \equiv \phi(\tilde{t}_k) \sim 7.756 M_P \qquad \sigma = \sqrt{2/3}\,\phi/M_P$$

$$n_{\mathrm{S}} \sim 0.968$$

$$r = 0.0028$$

$\tilde{t}_k$: 1st horizon crossing time for mode *k*

Inflation (2)

$$f(R) = R + \alpha R^2$$

[Starobinsky, Phys. Lett. B 91, 99 (1980)]

$$V(\Psi) = \frac{1}{8\alpha}\left(1 - e^{-\sqrt{2/3}\Psi}\right)^2$$

$$n_{\mathrm{S}} \simeq 1 - \frac{2}{N}, \qquad r = \frac{12}{N^2}$$

$$\Psi = \sqrt{\frac{3}{2}}\ln(1 + 2\alpha \mathrm{R})$$

$$8\pi G = 1$$

- $N = 50$ ⇨ $n_{\mathrm{S}} = 0.960$, $r = 0.00480$
- $N = 60$ ⇨ $n_{\mathrm{S}} = 0.967$, $r = 0.00333$

Cf. [Hinshaw *et al.*, Astrophys. J. Suppl. 208, 19 (2013)]

Quantum Correction

$$\Gamma^{(1)}_{\mathrm{Landau}} = \frac{1}{2}\mathcal{V} M_{\mathrm{P}}^2 \left(R + \frac{R^2}{6M^2}\right) + \frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\frac{1}{2}\Delta_0 - \frac{R}{2}\right)\right]$$

$$+\frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_1 - \frac{R}{4}\right)\right] - \ln\det\left[\frac{1}{\mu^2}\left(-\Delta_0 - \frac{R}{2} + M^2\right)\right]$$

$$-\frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_2 + \frac{R(R+12M^2)}{6(R+3M^2)}\right)\right]$$

$$\mathcal{V} = \frac{384\pi^2}{R^2}$$

Stability issue

$$\Gamma_{\text{on-shell}} = \frac{24\pi f_0}{GR^2} + \frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_0 + X_1\right)\right] + \frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_0 + X_2\right)\right]$$

$$-\frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_1 - \frac{R}{4}\right)\right] + \frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_2 + \frac{R}{6}\right)\right]$$

$$X_{1,2} = \frac{1}{2}\left(-\frac{a_1}{a_2} \pm \sqrt{\frac{a_1^2}{a_2^2} - \frac{4a_0}{a_2}}\right)$$

Condition to obtain two positive solution:

$$\frac{a_1}{a_2} < 0\ , \qquad \left(\frac{a_1}{a_2}\right)^2 \geq \frac{4a_0}{a_2} \geq 0$$

Quantizatin of the maximally symmetric（de Sitter) space

Euclidean action

$$I_{\mathrm{E}}[\tilde{g},\tilde{\phi}] = -\frac{1}{16\pi G}\int d^4x\,\sqrt{-\tilde{g}}\,f(\tilde{R},\tilde{K},\tilde{\phi})$$

$$R_{ijrs} = \frac{R}{12}\left(g_{ir}g_{js} - g_{is}g_{jr}\right), \quad R_{ij} = \frac{R}{4}\,g_{ij}\ , \qquad R = constant$$

g_{ij} : Metric of the maximally symmetric space

Fluctuations around the constant curvature solution

$$\begin{cases} \tilde{g}_{ij} = g_{ij} + h_{ij}\ , & |h_{ij}| \ll 1 \\ \tilde{g}^{ij} = g^{ij} - h^{ij} + h^{ik}h^{j}_{k} + \mathcal{O}(h^3)\,, & h = g^{ij}h_{ij} \\ \tilde{\phi} = \phi + \varphi\,, & |\varphi| \ll 1 \end{cases}$$

Euclidean action

Around the background fields $\{g_{ij}, \phi\}$**, we expand** $\sqrt{-\tilde{g}} f(\tilde{R}, \tilde{K}, \tilde{\phi})$ **.**

$$I_{\mathrm{E}}[g, \phi] \sim -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[f_0 + \mathcal{L}_1 + \underline{\mathcal{L}_2} \right]$$

Quantum correction

$$f_0 = f(R, K, \phi)$$

$$\mathcal{L}_1 = \frac{1}{4} X h + f_\phi \phi$$

$$X = 2 f_0 - R f_R$$

Euclidean action (2)

$$\mathcal{L}_2 = -\frac{1}{2} f_R h^i_k \nabla_i \nabla_j h^{jk} + \frac{1}{4} f_R h_{ij} \Delta h^{ij} - \frac{1}{24} R f_R h_{ij} h^{ij} + \frac{1}{2} f_R h \nabla_i \nabla_j h^{ij}$$

$$+ f_{R\phi} \varphi \nabla_i \nabla_j h^{ij} + \frac{1}{2} f_{RR} \nabla_i \nabla_j h^{ij} \nabla_r \nabla_s h^{rs} - f_{RR} \Delta h \nabla_i \nabla_j h^{ij} - \frac{1}{4} R f_{RR} h \nabla_i \nabla_j h^{ij}$$

$$-\frac{1}{48} R f_R h^2 - \frac{1}{2} f_K \varphi \Delta \varphi - \frac{1}{4} f_R h \Delta h - f_{R\phi} h \Delta \varphi + \frac{1}{4} R f_{RR} h \Delta h + \frac{1}{2} f_{\phi\phi} \varphi^2$$

$$-\frac{1}{4} R f_{R\phi} h \varphi + \frac{1}{32} R^2 f_{RR} h^2 + \frac{X}{16} (h^2 - 2 h_{ij} h^{ij}) + \frac{1}{2} f_\phi h \varphi$$

On-shell Lagrangian density: $X \to 0 \,, \; f_\phi \to 0$

Expansion of h_{ij}

$$h_{ij} = \hat{h}_{ij} + \nabla_i \xi_j + \nabla_j \xi_i + \nabla_i \nabla_j \sigma + \frac{1}{4} g_{ij} (h - \Delta_0 \sigma)$$

σ : Scalar component

ξ_i : Vector component

$\hat{h}_{ij}$: Tensor component

$$\nabla_i \xi^i = 0 \,, \qquad \nabla_i \hat{h}^i_j = 0 \,, \qquad \hat{h}^i_i = 0$$

Expression of $\mathcal{L}_2$

$$\begin{aligned}
\mathcal{L}_2 \;=\; & \frac{1}{32}\,\sigma\,\Big(9 f_{RR}\Delta\Delta\Delta\Delta - 3 f_R\Delta\Delta\Delta + 6 f_{RR}R\Delta\Delta\Delta \\
& \quad - f_R R\Delta\Delta + f_{RR}R^2\Delta\Delta - 3X\Delta\Delta - RX\Delta\Big)\,\sigma \\
& + \frac{1}{32}\,h\,\Big(9 f_{RR}\Delta\Delta - 3 f_R\Delta + 6 f_{RR}R\Delta - {f_R}^2 R + f_{RR}R^2 + X\Big)\,h \\
& + \frac{1}{16}\,h\,\Big(-9 f_{RR}\Delta\Delta\Delta + 9 f_R\Delta\Delta - 6 f_R\Delta\Delta - 6 f_{RR}R\Delta\Delta + f_R R\Delta - f_{RR}R^2\Delta\Big)\,\sigma \\
& + \frac{1}{2}\,\varphi\,\left(-f_K\Delta + f_{\phi\phi}\right)\,\varphi + \frac{1}{4}\,h\,\left(-3 f_{R\phi}\Delta + 2 f_\phi - f_{R\phi}R\right)\,\varphi \\
& + \frac{1}{4}\,\sigma\,\left(+3 f_{R\phi}\Delta\Delta + f_{R\phi}R\Delta\right)\,\varphi \\
& + \frac{1}{16}\,\xi_i\,\left(4X\Delta + 4RX\right)\,\xi^i + \frac{1}{24}\,\hat{h}_{ij}\,\left(6 f_R\Delta - f_R R - 3X\right)\,\hat{h}^{ij}
\end{aligned}$$

Lagrangian

Gauge condition

$$\chi_k = \nabla_j h^j_k - \frac{1+\rho}{4} \nabla_k h$$

ρ : Real parameter

Gauge fixing

$$\mathcal{L}_{gf} = \frac{1}{2} \chi^i G_{ij} \chi^j$$

$$G_{ij} = \gamma\, g_{ij} + \beta\, g_{ij} \Delta$$

$\gamma,\ \beta$: Constants

Lagrangian (2)

Ghost Lagrangian

[Buchbinder, Odintsov, Shapiro, *Effective action in quantum gravity* (1992)]

$$\mathcal{L}_{gh} = B^i \, G_{ik} \frac{\delta \, \chi^k}{\delta \, \varepsilon^j} C^j$$

C_k : Ghost vector

B_k : Anti ghost vector

$$\frac{\delta \, \chi^i}{\delta \, \varepsilon^j} = g_{ij} \, \Delta \, + R_{ij} + \frac{1-\rho}{2} \nabla_i \nabla_j \quad \longleftarrow \quad \delta \, h_{ij} = \nabla_i \varepsilon_j + \nabla_j \varepsilon_i$$

$$\Longrightarrow \quad \mathcal{L}_{gh} = B^i \, \left(\gamma \, H_{ij} + \beta \, \Delta \, H_{ij} \right) \, C^j$$

$$H_{ij} = g_{ij} \left(\Delta \, + \frac{R_0}{4} \right) + \frac{1-\rho}{2} \nabla_i \nabla_j$$

Lagrangian (3)

$$\mathcal{L}_{gf} = \frac{\gamma}{2}\Bigg[\xi^k\left(\Delta_1+\frac{R_0}{4}\right)^2\xi_k+\frac{3\rho}{8}h\left(\Delta_0+\frac{R_0}{3}\right)\Delta_0\sigma$$
$$-\frac{\rho^2}{16}h\,\Delta_0 h-\frac{9}{16}\sigma\left(\Delta_0+\frac{R_0}{3}\right)^2\Delta_0\sigma\Bigg]$$
$$+\frac{\beta}{2}\Bigg[\xi^k\left(\Delta_1+\frac{R_0}{4}\right)^2\Delta_1\xi_k+\frac{3\rho}{8}h\left(\Delta_0+\frac{R}{4}\right)\left(\Delta_0+\frac{R}{3}\right)\Delta_0\sigma$$
$$-\frac{\rho^2}{16}h\left(\Delta_0+\frac{R_0}{4}\right)\Delta_0 h-\frac{9}{16}\sigma\left(\Delta_0+\frac{R_0}{4}\right)\left(\Delta_0+\frac{R_0}{3}\right)^2\Delta_0\sigma\Bigg]$$

Δ_0 : Laplacian acting on scalars

Δ_1 : Laplacian acting on transverse-traceless vectors

Δ_2 : Laplacian acting on transverse-traceless tensors

Lagrangian (4)

$$\mathcal{L}_{gh} = \gamma \left\{ \hat{B}^i \left(\Delta_{\,1} + \frac{R_0}{4} \right) \hat{C}^j + \frac{\rho - 3}{2} b \left(\Delta_{\,0} - \frac{R_0}{\rho - 3} \right) \Delta_{\,0} c \right\}$$

$$+ \beta \left\{ \hat{B}^i \left(\Delta_{\,1} + \frac{R_0}{4} \right) \Delta_{\,1} \hat{C}^j + \frac{\rho - 3}{2} b \left(\Delta_{\,0} + \frac{R_0}{4} \right) \left(\Delta_{\,0} - \frac{R_0}{\rho - 3} \right) \Delta_{\,0} c \right\}$$

$$C_k \equiv \hat{C}_k + \nabla_k c \,, \qquad \nabla_k \hat{C}^k = 0$$

$$B_k \equiv \hat{B}_k + \nabla_k b \,, \qquad \nabla_k \hat{B}^k = 0$$

Lagrangian (5)

Total Lagrangian: $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{gf} + \mathcal{L}_{gh}$

$$Z^{(1)} = e^{-\Gamma^{(1)}} = (\det G_{ij})^{-1/2} \int D[h_{ij}] D[C_k] D[B^k] \exp\left(-\int d^4x \sqrt{g}\, \mathcal{L}\right)$$

$$= (\det G_{ij})^{-1/2} \det J_1^{-1} \det J_2^{1/2}$$

$$\times \int D[h] D[\hat{h}_{ij}] D[\xi^j] D[\sigma] D[\hat{C}_k] D[\hat{B}^k] D[c] D[b] \exp\left(-\int d^4x \sqrt{g}\, \mathcal{L}\right)$$

$$J_1 = \Delta_0 , \qquad J_2 = \left(\Delta_1 + \frac{R_0}{4}\right)\left(\Delta_0 + \frac{R_0}{3}\right)\Delta_0$$

$$\det G_{ij} = \text{const} \ \det\left(\Delta_1 + \frac{\gamma}{\beta}\right) \det\left(\Delta_0 + \frac{R_0}{4} + \frac{\gamma}{\beta}\right)$$

[Buchbinder, Odintsov and Shapiro, *Effective action in quantum gravity* (1992)]
[Fradkin and Tseytlin, Nucl. Phys. B234, 472 (1984)]

Effective action

- $\rho = 1, \quad \beta = 0$:

$$\Gamma = I_{\mathrm{E}}[g,\phi] + \Gamma^{(1)}\,, \qquad\qquad I_{\mathrm{E}}[g,\phi] = \frac{24\pi f_0}{GR^2}$$

$$\begin{aligned}
\Gamma^{(1)} \;\; = \;\; & \frac{1}{2}\ln\det\left(b_4\Delta_0^{\,4} + b_3\Delta_0^{\,3} + b_2\Delta_0^{\,2} + b_1\Delta_{\,0} + b_0\right) - \ln\det\left(-\Delta_{\,0} - \frac{R}{2}\right) \\
& + \frac{1}{2}\ln\det\left(-\Delta_{\,1} - \frac{R}{4} - \frac{X}{2\gamma}\right) - \ln\det\left(-\Delta_{\,1} - \frac{R}{4}\right) \\
& + \frac{1}{2}\ln\det\left(-\Delta_{\,2} + \frac{R}{6} + \frac{X}{2f_R}\right)
\end{aligned}$$

$b_k \;\; (k = 0, \cdots 4)$ consists of $f(R, K, \phi)$ and its derivatives.

Effective action (2)

- $X \to 0\,,\ f_\phi \to 0$:

$$\Gamma^{(1)}_{\text{on-shell}} = \frac{1}{2}\ln\det\left[\frac{1}{\mu^4}\left(a_2\Delta_0^2 + a_1\Delta_0 + a_0\right)\right]$$

$$-\frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_1 - \frac{R}{4}\right)\right] + \frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_2 + \frac{R}{6}\right)\right]$$

$a_k\ (k = 0, 1, 2)$ consists of $f(R, K, \phi)$ and its derivatives.

μ^2 : Renormalized parameter

Effective action (3)

- $\rho = 1,\ \beta = 0,\ \gamma = \infty$:

$$\Gamma^{(1)}_{\mathrm{Landau}} = \frac{1}{2}\ln\det\left[\frac{1}{\mu^8}\left(c_4\Delta_0^{\,4} + c_3\Delta_0^{\,3} + c_2\Delta_0^{\,2} + c_1\Delta_0 + c_0\right)\right]$$

$$- \frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_1 - \frac{R}{4}\right)\right] - \ln\det\left[\frac{1}{\mu^2}\left(-\Delta_0 - \frac{R}{2}\right)\right]$$

$$+ \frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_2 - \frac{R}{3} + \frac{f_0}{f_R}\right)\right]$$

$c_k\ (k = 0, \cdots, 4)$ consists of $f(R, K, \phi)$ and its derivatives.

Expression of coefficients

$$
\begin{aligned}
a_0 &= f_R\left[Rf_{R\phi}^2 + f_{\phi\phi}(f_R - Rf_{RR})\right] \\
a_1 &= f_R\left[f_K(Rf_{RR} - f_R) + f_{R\phi}^2 - 3f_{\phi\phi}f_{RR}\right] \\
a_2 &= 3f_K f_R f_{RR}
\end{aligned}
$$

$$
\begin{aligned}
b_0 = &\ \frac{4{f_\phi}^2 X}{\gamma} + 4{f_\phi}^2 R - \frac{4f_\phi f_{R\phi} RX}{\gamma} - 4f_\phi f_{R\phi} R^2 + \frac{f_{\phi\phi} f_R RX}{\gamma} \\
&+ f_{\phi\phi} f_R R^2 - \frac{f_{\phi\phi} f_{RR} R^2 X}{\gamma} - f_{\phi\phi} f_{RR} R^3 - \frac{f_{\phi\phi} X^2}{\gamma} - f_{\phi\phi} RX \\
&+ \frac{{f_{R\phi}}^2 R^2 X}{\gamma} + {f_{R\phi}}^2 R^3
\end{aligned}
$$

Expression of coefficients (2)

$$
\begin{aligned}
b_1 \;=\;& -\frac{f_K f_R R X}{\gamma} - f_K f_R R^2 + \frac{f_K f_{RR} R^2 X}{\gamma} + f_K f_{RR} R^3 \\
&+ \frac{f_K X^2}{\gamma} + f_K R X + \frac{4 {f_\phi}^2 f_R}{\gamma} - \frac{4 {f_\phi}^2 f_{RR} R}{\gamma} \\
&+ 12 {f_\phi}^2 - \frac{12 f_\phi f_{R\phi} X}{\gamma} - 20 f_\phi f_{R\phi} R + \frac{2 f_{\phi\phi} f_R X}{\gamma} + 4 f_{\phi\phi} f_R R \\
&- \frac{5 f_{\phi\phi} f_{RR} R X}{\gamma} - 7 f_{\phi\phi} f_{RR} R^2 - 2 f_{\phi\phi} X + \frac{5 {f_{R\phi}}^2 R X}{\gamma} + 7 {f_{R\phi}}^2 R^2 \\
b_2 \;=\;& -\frac{2 f_K f_R X}{\gamma} - 4 f_K f_R R + \frac{5 f_K f_{RR} R X}{\gamma} + 7 f_K f_{RR} R^2 \\
&+ 2 f_K X - \frac{12 {f_\phi}^2 f_{RR}}{\gamma} - 24 f_\phi f_{R\phi} + 4 f_{\phi\phi} f_R - \frac{6 f_{\phi\phi} f_{RR} X}{\gamma} \\
&- 16 f_{\phi\phi} f_{RR} R + \frac{6 {f_{R\phi}}^2 X}{\gamma} + 16 {f_{R\phi}}^2 R
\end{aligned}
$$

Expression of coefficients (3)

$$
\begin{aligned}
b_3 &= -4f_K f_R + \frac{6f_K f_{RR} X}{\gamma} + 16 f_K f_{RR} R - 12 f_{\phi\phi} f_{RR} + 12 {f_{R\phi}}^2 \\
b_4 &= 12 f_K f_{RR}
\end{aligned}
$$

$$
\begin{aligned}
c_0 &= R\left[f_{\phi\phi}(2Rf_R - 2f_0 - R^2 f_{RR}) + (2f_\phi - Rf_{R\phi})^2\right] \\
c_1 &= f_0\,(2f_K R - 4f_{\phi\phi}) + 12{f_\phi}^2 - 20 f_\phi f_{R\phi} R \\
&\quad + R\left(-2f_K f_R R + f_K f_{RR} R^2 + 6 f_{\phi\phi} f_R - 7 f_{\phi\phi} f_{RR} R + 7{f_{R\phi}}^2 R\right) \\
c_2 &= 4 f_0 f_K - 6 f_K f_R R + 7 f_K f_{RR} R^2 - 24 f_\phi f_{R\phi} \\
&\quad + 4 f_{\phi\phi}\,(f_R - 4 f_{RR} R) + 16 {f_{R\phi}}^2 R \\
c_3 &= -4\left(f_K\,(f_R - 4f_{RR}R) + 3 f_{\phi\phi} f_{RR} - 3{f_{R\phi}}^2\right) \\
c_4 &= 12 f_K f_{RR}
\end{aligned}
$$

$f(R)$ gravity

$$\Gamma^{(1)}_{\text{on-shell}} = \frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-f_{RR}\left(\Delta_0+\frac{R}{3}\right)+\frac{f_R}{3}\right)\right] - \frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_1-\frac{R}{4}\right)\right]+\frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_2+\frac{R}{6}\right)\right]$$

$$\Gamma^{(1)}_{\text{Landau}} = \frac{1}{2}\ln\det\left[\frac{1}{\mu^4}\left(f_{RR}(6\Delta_0^2+5R\Delta_0-2f_R(\Delta_0+R)+2f_0\right)\right] - \frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_1-\frac{R}{4}\right)\right]-\frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_0-\frac{R}{2}\right)\right] + \frac{1}{2}\ln\det\left[\frac{1}{\mu^2}\left(-\Delta_2-\frac{R}{3}+\frac{f_0}{f_R}\right)\right]$$